

“LET Δ BE A COHEN-MACAULAY COMPLEX ...”

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Dedicated to Richard Stanley on the occasion of his 70th birthday



ABSTRACT. The concept of Cohen-Macaulay complexes emerged in the mid-1970s and swiftly became the focal point of an attractive and richly connected new area of mathematics, at the crossroads of combinatorics, commutative algebra and topology. As the main architect of these developments, Richard Stanley has made fundamental contributions over many years.

This paper contains some brief mathematical discussions related to the Cohen-Macaulay property, and some personal memories. The characterization of Gorenstein* and homotopy Gorenstein* complexes and the relevance in that connection of the Poincaré conjecture is discussed. Another topic is combinatorial aspects of a recent result on the homotopy Cohen-Macaulayness of certain subsets of geometric lattices, motivated by questions in tropical geometry.

1. INTRODUCTION

As is well known, Richard has a very good sense of humor. One of his favorite jokes around 1980 went somewhat like this:

Based on a talk given at the “Stanley@70” birthday conference”, MIT, June 2014.

“Can you imagine that at some time in the future it will be possible to begin a general math lecture with “Let Δ be a Cohen-Macaulay complex . . .” and go on from there without further explanation.”

The fact that this would not work as a joke today testifies to the emergence of this concept from what was then a remote corner of combinatorics into mainstream mathematics.

The concept of Cohen-Macaulay complexes arose in the mid-1970s from the work of Melvin Hochster [Hoc72, Hoc77], Gerald Reisner [Rei76], Richard Stanley [Sta75, Sta77], and (for the case of posets) Kenneth Baclawski [Bac76, Bac80]. Others soon followed and an attractive new area of mathematics took shape. The theory of Cohen-Macaulay complexes has applications to several concepts in combinatorics, such as matroids, polytope boundaries, geometric lattices, buildings, intersection lattices of hyperplane arrangements, and more.

Much of the appeal of the concept of CM-ness stems from its “interdisciplinary” character, building on and having bearing on several mathematical areas in addition to combinatorics, notably several parts of algebra, geometry and topology. However, what bestows lasting importance on the concept is its remarkable record of being a key ingredient both for “abstract” theoretical understanding and for “concrete” problem solving in several of these diverse areas.

As the main architect of these developments, Richard has made fundamental contributions over many years. His proof in 1975 of the Upper Bound Conjecture for spheres [Sta75] catapulted Cohen-Macaulay complexes into the limelight. Then his 1977 paper [Sta77] outlined the contours of a theory with many beautiful results and many appealing problems. His contributions to this area have been essential for the development of algebraic combinatorics, and have had a significant impact also on a wider mathematical territory, particularly for ring theory.

In this paper I meander among basic facts, brief mathematical discussions, and personal memories. One topic discussed is the topological characterization of homotopy Gorenstein* complexes and the relevance of the Poincaré conjecture in that connection, observed a few years ago by P. Hersh and the author (Section 4). Another topic is a recent result of K. Adiprasito and the author on the Cohen-Macaulayness of certain subsets of geometric lattices (Section 3), motivated by questions in tropical geometry. This result can be illustrated¹ by the following small “story”.

Suppose that each street corner of a crime-ridden city has been given a ranking number, reflecting how safe it is to visit that corner. To normalize the grading, the average rank has been set to be zero. Street corners with positive rank are considered safe, those with negative rank are not. Furthermore, a street in the city is considered safe if the average rank of all street corners along that street is positive, otherwise it is dangerous. The question is: Is it possible to walk from any safe street corner to any other safe one without ever passing a dangerous corner or walking along a dangerous street? For the answer, see Section 3.3.

¹in the sense of Diaconis [Dia14].

2. COHEN-MACAULAY COMPLEXES REVISITED

The purpose of this section is to remind the reader of some definitions and basic facts.

2.1. Simplicial complexes everywhere. We begin with a few words to fix notation and agree on basic definitions.

A *simplicial complex* (or *abstract simplicial complex*, or just *complex*) is a finite set V together with a family Δ of subsets of V such that $A \subseteq B \in \Delta$ implies that $A \in \Delta$. The elements of V are the *vertices* and the members of Δ the *faces* of the complex. We assume that the empty set is a face. The nonempty faces are the *proper* faces. The vertices are usually clear from context, and if so we use only Δ to denote the complex.

The *dimension* of a face is one less than its cardinality, and the dimension of Δ is the maximal dimension of any of its faces.

A mathematical concept could hardly be simpler than that of a simplicial complex. All that is required is a family of subsets of a finite set, closed under the operation of taking subsets. Simplicial complexes arise everywhere in combinatorics under different names: hypergraphs, hereditary set families, order ideals in the Boolean lattice, etc.

To define a Cohen-Macaulay complex is not as elementary. There are two ways to proceed, equivalent as it turns out, via algebra (commutative rings) or via topology (simplicial homology).

2.2. Cohen-Macaulay complexes via commutative rings. In commutative algebra simplicial complexes correspond to squarefree monomial ideals. In the following, \mathbf{k} denotes a field or the ring \mathbb{Z} of integers. Suppose that Δ is a complex on vertex set $V = \{1, 2, \dots, n\}$. Let $A = \mathbf{k}[x_1, x_2, \dots, x_n]$ and let I_Δ be the ideal generated by squarefree monomials $x_{i_1}x_{i_2} \cdots x_{i_k}$ such that $\{i_1, i_2, \dots, i_k\} \notin \Delta$. The ideal I_Δ is called the *Stanley-Reisner ideal* and the ring $\mathbf{k}[\Delta] = A/I_\Delta$ the *Stanley-Reisner ring*.

Let

$$0 \rightarrow F_j \rightarrow F_{j-1} \rightarrow \cdots \rightarrow F_0 \rightarrow A/I_\Delta \rightarrow 0$$

be a minimal free resolution of $\mathbf{k}[\Delta]$ as an A -module. We know from Hilbert's syzygy theorem that its length j satisfies $j \leq n$, and from the Auslander-Buchsbaum theorem that $n - d \leq j$, where $d \stackrel{\text{def}}{=} \dim \Delta + 1$. The integer $n - j$ is called the *depth* of $\mathbf{k}[\Delta]$, and the rank of the free module F_j is the *type* of $\mathbf{k}[\Delta]$. Clearly, $\text{depth} \leq d$.

Here are the basic definitions, for the ring $\mathbf{k}[\Delta]$ and for the complex Δ .

- Definition 2.1.** (i) $\mathbf{k}[\Delta]$ is Cohen-Macaulay if its depth equals d .
(ii) $\mathbf{k}[\Delta]$ is Gorenstein if it is a Cohen-Macaulay ring of type 1.
(iii) Δ is Cohen-Macaulay, resp. Gorenstein, if $\mathbf{k}[\Delta]$ is.
(iv) Δ is Gorenstein* if it is Gorenstein and not acyclic over \mathbf{k} .

In the sequel we consider only Gorenstein* complexes, the reason being that a general Gorenstein complex is a multiple cone over a Gorenstein* complex, and the latter carries all relevant information.

2.3. Cohen-Macaulay complexes via simplicial homology. In topology simplicial complexes play the important role of encoding compact topological spaces. With a complex Δ is associated its *geometric realization* $||\Delta||$. Encoding reasonably nice compact spaces via triangulation has always been an important tool in topology. This importance has increased in recent years as triangulation is a necessary part of protocols for communicating with computers about topological spaces.

If F is a face of Δ , then

$$\text{link}_\Delta(F) \stackrel{\text{def}}{=} \{G \in \Delta \mid G \cup F \in \Delta \text{ and } G \cap F = \emptyset\}$$

is a subcomplex, called the *link of Δ at F* . It carries local information about the complex and its geometric realization. Note that $\text{link}_\Delta(\emptyset) = \Delta$. We say that $\text{link}_\Delta(F)$ is *proper* if $F \neq \emptyset$.

The following topological characterizations of Cohen-Macaulayness, due to Reisner [Rei76], is a corner stone of the theory.

Theorem 2.2. *A simplicial complex Δ is Cohen-Macaulay over \mathbf{k}*

$$\iff \tilde{H}_i(\text{link}_\Delta(F); \mathbf{k}) = 0 \text{ for all } F \in \Delta \text{ and all } i < \dim(\text{link}_\Delta F).$$

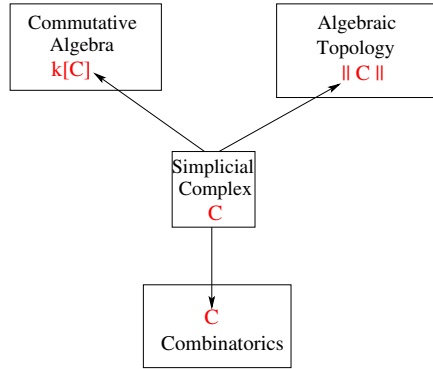


FIGURE 2.1. Simplicial complexes provide the link

2.4. Commutative Algebra \leftrightarrow Algebraic Topology. At the core of Stanley-Reisner theory stands the discovery, well exemplified by Reisner's Theorem [Rei76], that some central properties of commutative rings correspond to important topological properties of simplicial complexes. The following formula, due to Hochster [Hoc77], gives a particularly beautiful instance of this correspondence.

$$(2.1) \quad \beta_{i,j}(\mathbf{k}[\Delta]) = \sum_{E \subseteq V: |E|=j} \dim_{\mathbf{k}} \tilde{H}_{j-i-1}(\Delta|_E; \mathbf{k})$$

Here on the left hand side are the doubly indexed ring-theoretic Betti numbers, which record the dimensions of the resolvants in a minimal free resolution together with information about shifting of degrees. On the right hand side are the topological Betti numbers of simplicial reduced homology of subcomplexes of Δ induced on subsets E of the vertices.

A very interesting fact is that the ring-theoretic properties of being Cohen-Macaulay or Gorenstein* over \mathbf{k} are topological properties; they depend only on the homeomorphism type of the space $||\Delta||$. On the other hand, as pointed out by Reisner [Rei76] for the Cohen-Macaulay case, these properties depend on field characteristic. For instance, triangulations of the real projective plane $\mathbb{R}P^2$ are Cohen-Macaulay over a field \mathbf{k} if and only if $\text{char}(\mathbf{k}) \neq 2$. Similarly, triangulations of real projective 3-space $\mathbb{R}P^3$ are Gorenstein* if $\text{char}(\mathbf{k}) \neq 2$, but are not even Cohen-Macaulay if $\mathbf{k} = \mathbb{Z}$ or \mathbf{k} is a field of characteristic 2.

2.5. The homotopy Cohen-Macaulay property. We now strengthen the concept of Cohen-Macaulayness by replacing the vanishing of homology groups in the definition by the vanishing of corresponding homotopy groups π_i .

Definition 2.3. *A simplicial complex Δ is homotopy Cohen-Macaulay if $\pi_i(\text{link}_\Delta F) = 0$ for all faces F and all $i < \dim(\text{link}_\Delta F)$.*

By basic algebraic topology we have the following characterization.

Proposition 2.4. *A simplicial complex Δ is homotopy Cohen-Macaulay*

$$\iff \begin{cases} \Delta \text{ is Cohen-Macaulay over } \mathbb{Z}, \text{ and} \\ \text{all links of dimension } \geq 2 \text{ are simply connected.} \end{cases}$$

The concept of homotopy Cohen-Macaulayness first appeared in Quillen's paper [Qui78] on p -subgroups. He offered this motivation: "we use the stronger definition so that our theorems are in their best form".

In spite of being extremely natural from a topological point of view, this property is itself not topologically invariant. This is witnessed by the 5-dimensional sphere, which in addition to its standard triangulations, e.g. as the boundary of a 6-simplex, admits triangulations which are not homotopy Cohen-Macaulay [Edw75].

Over the years many properties related to Cohen-Macaulayness have been introduced and studied. Figure 2.2 shows some of the most important such properties, ordered as a poset with logical implication arrows directed down.

This is not the place to enter a technical discussion or attempt any kind of survey. As was stated initially, all that this section wants to convey are some brief reminders and comments about the basics.

3. COHEN-MACAULAYNESS OF FILTERED GEOMETRIC LATTICES

In this section we discuss a recent result motivated by questions in tropical geometry. It concerns the Cohen-Macaulayness of $\Delta(P)$ for certain subsets P of geometric lattices. Here the simplicial complex $\Delta(P)$ associated to a poset P is the collection of its chains (totally ordered subsets). We do not distinguish notationally between a poset P and its order complex $\Delta(P)$.

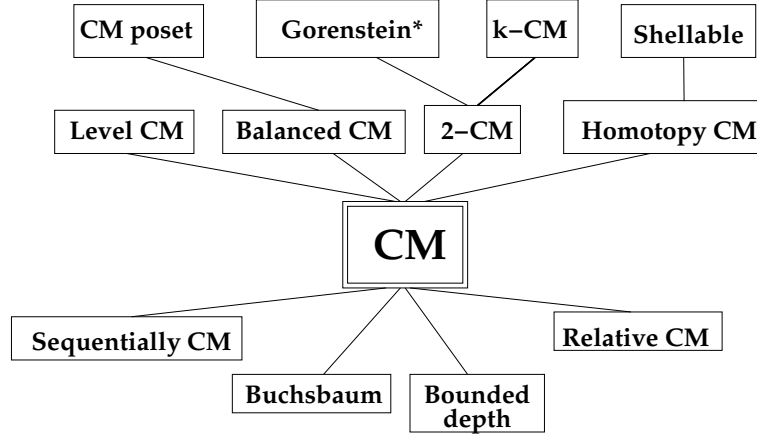


FIGURE 2.2. Properties of complexes related to CMness.

One of the properties of Cohen-Macaulayness that makes it so useful is its *resilience* – several useful constructions on complexes preserve the Cohen-Macaulay property. A good example is so called “rank-selection”, by which is meant the removal of all elements of specified rank-levels in a Cohen-Macaulay poset. This operation preserves Cohen-Macaulayness, as was shown in varying degrees of generality by Baclawski, Munkres, Stanley and Walker.

By comparison, the class of Gorenstein* complexes is not at all resilient. No removal of vertices leads to a Gorenstein* complex of the same dimension.

3.1. Geometric lattices. A finite lattice is called *geometric* if it is *semimodular* and *atomistic*. A geometric lattice is pure, so it has a well-defined rank function, $\rho : L \rightarrow \mathbb{Z}^+$. Being semimodular means that

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$$

for all $x, y \in L$, and being atomistic means that every element in the lattice is a join of atoms (atoms being the elements that cover $\hat{0}$). Geometric lattices are cryptomorphic to matroids, they arise as the lattice of closed sets of a matroid, see [Oxl11].

The order complex of a geometric lattice was one of the first examples of a Cohen-Macaulay complex. This came about in the following way. In his influential paper on the combinatorial Möbius function [Rot64], Rota proved that the coefficients of the characteristic polynomial

$$\chi(\mathcal{L}; z) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{L}} \mu(\hat{0}, x) z^{r-\rho(x)}$$

of a geometric lattice \mathcal{L} alternate in sign:

$$\chi(\mathcal{L}; z) = z^r - a_{r-1}z^{r-1} + a_{r-2}z^{r-2} - a_{r-3}z^{r-3} + a_{r-4}z^{r-4} - \dots$$

For the special case of chromatic polynomials of graphs this was a well-known phenomenon. The sign alternation is implied by the stronger set of inequalities

$$(-1)^{\rho(x)} \mu_{\mathcal{L}}(\hat{0}, x) > 0$$

valid for each $x \in \mathcal{L}$.

Rota's paper, which deals with the Euler characteristic of order complexes, hinted at a homological explanation of the result. This was worked out in a follow-up paper by Folkman [Fol66], who explicitly determined the homology of a geometric lattice. He showed that there is non-vanishing homology only in the top dimension. Since intervals in a geometric lattice are themselves geometric lattices, Cohen-Macaulayness can be deduced.

3.2. Filtered geometric lattices. The resilience of the Cohen-Macaulay property is well exemplified on the class of geometric lattices. For instance, it is known since long that Cohen-Macaulayness and dimension are preserved by removal of any chain, and of certain antichains, from a geometric lattice [Bjö80, Bac82], and also by removal of any principal filter [WW86]. The following is a recent discovery in this vein.

Let \mathcal{L} be a geometric lattice of rank r and with set of atoms A , and let $\omega : A \rightarrow \mathbb{R}$ be a real-valued function assigning a number, called a *weight*, to each atom. Extend the weight function to all $X \in \mathcal{L}$ by summation: $\omega(X) = \sum_{a \in X} \omega(a)$. We assume that ω is *generic*, meaning that $\emptyset \neq X \neq Y \neq \emptyset \Rightarrow \omega(X) \neq \omega(Y)$.

For $t \in \mathbb{R}$, let $\mathcal{L}^{>t} \stackrel{\text{def}}{=} \{X \in \mathcal{L} \mid \omega(X) > t\}$ with the induced partial order. We call posets of this form *filtered geometric lattices*. They do not need to be lattices. The following result was proved by Adiprasito and the author [AB14].

Theorem 3.1. *Suppose that $t \leq \min[0, \omega(A)]$. Then, $\mathcal{L}^{>t}$ is homotopy Cohen-Macaulay and of the same dimension.*

The theorem is proved by techniques from topological poset theory such as lexicographic shellability and Quillen-type fiber arguments. It was conjectured by Michalkin and Ziegler [MZ08] in a stronger form, namely that $\mathcal{L}^{>t}$ itself is shellable. This claim is still open. The reason for their conjecture is that it implies information that is crucial for proving Lefschetz-type section theorems in tropical geometry. We will not here pursue this path into the territory of tropical geometry.

From now on we specialize to the conditions $t = \omega(A) = 0$. Then, with respect to the weights ω and $-\omega$, we have that \mathcal{L} splits into two parts:

$$\mathcal{L}^+ \stackrel{\text{def}}{=} \mathcal{L}^{>0} \quad \text{and} \quad \mathcal{L}^- \stackrel{\text{def}}{=} \mathcal{L}^{<0}$$

Both parts are homotopy Cohen-Macaulay and of the same dimension as \mathcal{L} .

All properties of geometric lattices that depend only on Cohen-Macaulayness can now automatically be generalized to filtered geometric lattices. For example, we get filtered characteristic polynomials,

$$\chi(\mathcal{L}^+; z) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{L}^+} \mu_{\mathcal{L}^+}(\hat{0}, x) z^{r-\rho(x)}.$$

whose coefficients alternate in sign:

$$\chi(\mathcal{L}^+; z) = z^r - b_{r-1}z^{r-1} + b_{r-2}z^{r-2} - b_{r-3}z^{r-3} + b_{r-4}z^{r-4} - \dots$$

For instance, the weighted configuration in Figure 3.2 has these filtered characteristic polynomials:

$$\chi(\mathcal{L}^+; z) = z^3 - 4z^2 + 3z \text{ and } \chi(\mathcal{L}^-; z) = z^3 - 2z^2 + z.$$

The sign alternation is implied by

$$(-1)^{\rho(x)} \mu_{\mathcal{L}^+}(\hat{0}, x) \geq 0,$$

which is valid for each $x \in \mathcal{L}$ as a consequence of Cohen-Macaulayness.

As a special case, putting generic weights on the edges of a graph we are led to consider “filtered chromatic polynomials”. Such polynomials have to our knowledge not been explored.

3.3. Combinatorial aspects. We will now have a look at what Theorem 3.1 says in the first non-trivial case; that of rank 3. The proper part of a geometric lattice of rank 3 is a bipartite graph, and for graphs, Cohen-Macaulayness just means being connected. This connectivity result can be illustrated in the following concrete way for matroids representable over \mathbb{R} .

Consider a finite collection of points in the plane \mathbb{R}^2 . Along with the points we consider also the lines that are spanned by subsets of the points, see Figure 3.1.

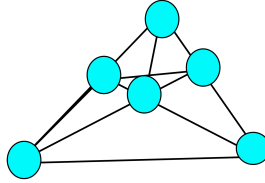


FIGURE 3.1. A configuration of 6 points spanning 7 lines in \mathbb{R}^2 .

A real number, called its *weight*, is assigned to each point, and we assume that these numbers sum to zero, but are otherwise generic (Figure 3.2).

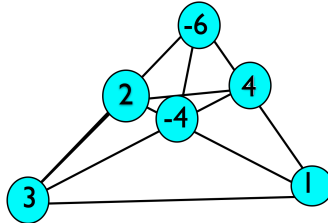


FIGURE 3.2. The weighted point configuration.

By the weight of a line we mean the sum of the weights of its points. With this we can distinguish between positive and negative lines. For instance, in our running example we have four negative lines $\{1, -4, 2\}$, $\{1, 4, -6\}$, $\{2, 3, -6\}$, $\{-4, -6\}$. This leaves three positive lines (Figure 3.3).

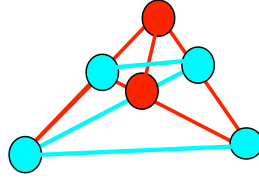


FIGURE 3.3. Negative points and lines are marked red.

The question now is this: Is it possible to walk in the configuration from any positive point to any other positive point, never visiting a negative point or walking along a negative line?

The answer is YES, since, as predicted by the theorem, the configuration of positive points and lines is connected (Figure 3.4).

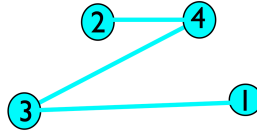


FIGURE 3.4. The positive part is connected.

It is natural to wonder how long such a walk can be in the worst case? Consider for a weighted configuration of n points the graph whose vertices are the positive points and whose edges are the pairs of positive points that span a positive line. What is the maximal diameter of such a graph?

The theorem contains similar geometric-combinatorial information also for higher ranks. For instance, fix integers $1 < k < r$ and consider a configuration of n points in \mathbb{R}^{r-2} . A real number weight is assigned to each point, and we assume that these otherwise generic numbers sum to zero. By summation each flat of rank k receives a weight, so there are positive and negative k -flats.

The theorem guarantees that it is possible to walk in the configuration from any positive point to any other positive point, never visiting a negative point or moving across a negative k -flat.

The case $k = 2$ gives us again the positive point–line graphs, but now in Euclidean space of arbitrarily high dimension \mathbb{R}^{r-2} . Again, the diameter question can be asked.

4. GORENSTEIN* COMPLEXES'

4.1. The Gorenstein* property. The basic topological characterization of Gorenstein* complexes is the following, derived by Stanley [Sta77] from the work of Hochster [Hoc77].

Theorem 4.1. *A simplicial complex Δ is Gorenstein* over \mathbf{k}*

$$\iff \tilde{H}_i(\text{link}_\Delta(F); \mathbf{k}) = \begin{cases} \mathbf{k}, & \text{for all } F \in \Delta \text{ and } i = \dim(\text{link}_\Delta F), \\ 0 & \text{for all } F \in \Delta \text{ and all } i < \dim(\text{link}_\Delta F). \end{cases}$$

For some alternative criteria, see [Sta96, Theorem 5.1]. The following is a slightly sharper version of criterion (d) of that theorem.

Theorem 4.2. *Let Δ be a simplicial complex, and $\mathbf{k} = \mathbb{Z}$ or a field. Then Δ is Gorenstein* over \mathbf{k} if and only if*

- (1) Δ is Cohen-Macaulay over \mathbf{k} , and
- (2) Δ is thin².

If \mathbf{k} is a field of char $\neq 2$ we also need a third condition:

- (3) $\tilde{\chi}(\Delta) = (-1)^{\dim \Delta}$.

Here is a sketch of the proof. The necessity of conditions (1), (2) and (3) is easy to deduce from Theorem 4.1. To prove sufficiency, the first observation to be made is that, in case $\mathbf{k} = \mathbb{Z}$ or a field of char $= 2$, condition (3) on the reduced Euler characteristic $\tilde{\chi}$ is implied by (1) and (2). So, for the proof of sufficiency condition (3) can be assumed in all cases.

The second observation is that Cohen-Macaulayness implies being dually connected, so being Cohen-Macaulay and thin implies being a pseudomanifold. The two top-dimensional homology groups of a pseudomanifold are well known, see e.g. Munkres [Mun84]. Using this and verifying for all links finishes the proof.

4.2. The homotopy Gorenstein* property. Just like for Cohen-Macaulay complexes, the Gorenstein* property has a homotopy version. Somewhat surprisingly, this property turns out to be essentially equivalent to being a PL sphere, as shown by Theorem 4.5 below.

Proposition 4.3. *Δ is homotopy Gorenstein* $\Leftrightarrow \Delta$ is homotopy Cohen-Macaulay and thin.*

Proof. We know from Theorem 4.2 that

$$\text{Gorenstein* over } \mathbb{Z} \iff \text{Cohen-Macaulay over } \mathbb{Z} \text{ and thin.}$$

Adding the condition “and all links of dimension ≥ 2 are simply-connected” to both sides changes this equivalence to

$$\text{homotopy Gorenstein*} \iff \text{homotopy Cohen-Macaulay and thin.}$$

□

²A pure complex is said to be *thin* if every face of codimension one lies in exactly two maximal faces. It is *dually connected* if one can walk from any maximal face to any other one via steps across codimension one faces. It is a *pseudomanifold* if it is thin and dually connected.

The crucial step in the following proof requires an application of the PC-theorem, by which we mean the theorem of Smale, Freedman and Perelman verifying the generalised Poincaré conjecture in all dimensions. Theorem 4.4 and the usefulness of the PC-theorem for its proof was discovered a few years ago independently by P. Hersh [Her10] and the author (unpublished).

Theorem 4.4. Δ is homotopy Gorenstein* $\Rightarrow \Delta$ is homeomorphic to a sphere.

Proof. The theorem is proved by induction on dimension, using that

$$\text{if } F \subseteq G \in \Delta, \text{ then } \text{link}_{\text{link}_\Delta(F)}(G \setminus F) = \text{link}_\Delta(G).$$

The claim is certainly true in low dimensions, for instance if $\dim \Delta = 1$ then Δ must be a 1-sphere.

By definition, for Δ to be homotopy Gorenstein* means that the link at every face $F \in \Delta$ has the homotopy type of a $(\dim \Delta - 1 - \dim F)$ -sphere. Hence by induction, $\text{link}_\Delta(F)$ is homeomorphic to a sphere of the appropriate dimension for all proper faces $F \in \Delta$. This shows that $||\Delta||$ is a topological manifold. Also $\Delta = \text{link}_\Delta(\emptyset)$ is by assumption a simply-connected homology sphere. The PC-theorem therefore implies that $||\Delta||$ is homeomorphic to a sphere. \square

The inductive procedure of the preceding proof can be sharpened in many cases, leading to spheres in the piecewise linear category³. The obstruction to being able to say “in all cases” is the possibility that non-PL 4-spheres might turn up among the links. Defining away this possibility, we reach this result.

Theorem 4.5. Let Δ be a simplicial complex. Then the following conditions are equivalent:

- (1) Δ is homotopy Gorenstein* and every link of dimension 4 is a PL sphere
- (2) Δ is a PL sphere.

Proof. The proof that (1) implies (2) is via the same inductive procedure, now however keeping track of combinatorial and PL spheres. For the inductive step, we have that all proper links in Δ are PL spheres. That means that Δ itself is a combinatorial sphere, and hence PL.

In the other direction, in a PL sphere every link of a face is itself a PL sphere, and hence in particular a homotopy sphere. \square

We conclude with a list showing the place of the homotopy Gorenstein* property in a hierarchy of near-spherical complexes.

Theorem 4.6. Let Δ be pure. In the following list, each property implies its successor.

³A simplicial complex which is piecewise linearly homeomorphic to the boundary of a simplex is called a *PL sphere*. A *combinatorial manifold* is a triangulation of a topological manifold such that the link at every vertex is a PL sphere. A combinatorial manifold which is homeomorphic to a sphere is called a *combinatorial sphere*. PL spheres are combinatorial.

For dimensions $d \neq 4$, a triangulation of the d -sphere is PL if and only if it is combinatorial. This follows from major work in topology showing that in these dimensions there is a unique PL structure for spheres. For $d = 4$ the question whether a combinatorial sphere must be PL is open.

- (a) Δ is thin and shellable
- (b) Δ is a PL sphere
- (c) Δ is homotopy Gorenstein*
- (d) Δ is homeomorphic to a sphere
- (e) Δ is Gorenstein* over \mathbb{Z}
- (f) Δ is Gorenstein* over \mathbf{k} , for some field \mathbf{k}
- (g) Δ is a pseudomanifold and $\tilde{\chi}(\Delta) = (-1)^{\dim(\Delta)}$
- (h) Δ is thin

Implications in this list that have not been discussed in this section are either well known or else elementary in view of well known facts. All the implications in Theorem 4.6 are strict, except possibly (b) \Rightarrow (c).

5. SOME PERSONAL RECOLLECTIONS

I first met Richard at a conference in Berlin in 1976. He gave a series of two hour-long talks with title “Cohen-Macaulay Complexes”. After that I was hooked. My main interest was and is in combinatorics, but with strong side interests in algebra and topology. I was struck by the beauty and power of the new area outlined by Richard.

Then I spent the academic year 1977–1978 at MIT, which deepened my interest and increased my knowledge. Richard was, as always, very generous with discussions and advice. Among a multitude of memories from those days, let me mention a couple of minor observations which seemed a bit puzzling to me at the time but later had their explanations.

One was that Richard’s desk seemed to be stuffed with packs of sorted index cards. Often in connection with discussions he would retrieve an index card and make some annotations. I understood later that this was part of a systematic gathering of material for his future books EC1 and EC2, particularly for the exercises. At some point I observed that a copy of a Springer Lecture Notes volume with the mysterious title “Toroidal Embeddings” ([KKMSD73]) had been lying on his desk for a while. I asked Richard what the yellow book was all about and why he was reading it. At that time I had never heard of toric varieties, or of any algebraic-geometric aspect of convex polytopes, so I was quite amazed by his answer. He said that he was convinced that the machinery surrounding these varieties had ingredients that would one day add up to prove the necessity part of the g -conjecture for simplicial polytopes (a conjectured characterization of f -vectors of simplicial polytopes). As we all know, he was right [Sta80].

Now, move fast forward to the spring of 1981. Institut Mittag-Leffler had a research program then in “Commutative algebra”, of which Richard was a participant. During his time in Stockholm Richard gave a series of eight two-hour lectures on “Combinatorics and Commutative Algebra” at Stockholm University, giving a splendid overview of the new connections that had been discovered in recent years. I had the benefit of taking notes at the lectures and writing up what became the core of Richard’s “green book” [Sta83]. I learned a lot from connecting the dots as Richard expertly and sometimes rapidly moved from topic to topic.

It so happened that the first Nordic Combinatorics Conference took place in Norway at the time when Richard was in Stockholm. The venue was Utstein Abbey, an old monastery on an island in the North Sea, just off the Norwegian coast. We travelled there with a group of mathematicians by train and boat from Stockholm, a journey that involved taking night train from Oslo to Stavanger. The meeting, including frequent refreshment sessions, was graciously hosted by Prof. Selmer from Bergen.

For the return trip Richard and I took a different route from the others. Wanting to experience more of the majestic Norwegian coast, we took a local boat north, hopping up the coast to Bergen. After a night there we continued by train over the mountains to Oslo, and then on to Stockholm.

There were many hours spent travelling and most of this time we discussed mathematics. Somehow the idea to write a book took shape. Richard had a lot of material and ideas beyond what appeared in his Stockholm lectures (or, the first edition of the green book). The plan was that Richard would write about algebraic matters and I would be in charge of the topological stuff. At that point, I had since a few years taken great interest in topological methods in combinatorics and poset topology, an area for which there was much nice material but little exposition available at the time.

As fjords and glaciers passed by outside we worked out more and more detailed plans, chapter by chapter, leading to the following table of contents.

Ring-theoretic and combinatorial aspects of simplicial complexes

Table of contents:

1. *Topological preliminaries*
2. *Algebraic preliminaries*
3. *Cohen-Macaulay complexes*
4. *Shellable complexes*
5. *k -Cohen-Macaulay complexes*
6. *Buchsbaum complexes*
7. *Gorenstein complexes*
8. *Examples*
9. *Multigradings*
10. *Local cohomology*
11. *Canonical modules*
12. *Balanced complexes*
13. *Bounds*
14. *Posets*
15. *Topological methods for posets*
16. *Group actions*
17. *Further developments*

As the train finally reached Stockholm we went our separate ways. No further work on this book project was ever done. All that remains is a dusty folder with some notes from our discussions en route from Norway, and (at least on my side) some fond memories.

Acknowledgment: The pen sketch on the title page is due to artist Berta Hansson. It was drawn during the author's thesis defense at Stockholm University in May 1979, at which Richard Stanley served as opponent

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